Chapter 3

Amplifiers

For low frequency amplifier design, a transistor can be modeled using an equivalent circuit. At microwave frequencies, reflections from the input and output ports become important, so a network description characterized by S-parameters as a function of frequency is required. The key parameters of a transistor for microwave amplification are

- Gain as a function of frequency
- $f_T =$ frequency at which gain drops to unity
- S-parameters as a function of frequency
- Noise figure: characterizes noise added to signal by the device ($\mathrm{SNR}_{\mathrm{out}} < \mathrm{SNR}_{\mathrm{in}}$).

f_T determines the usable bandwidth of the transistor, the S-parameters are used to design matching networks to match to the transistor and determine the gain and stability of the amplifier, and the noise figure is used to determine the amount of noise produced by the transistor.

3.1 Gain

The most important quantity for a microwave amplifier is power gain. In order to analyze amplifier gain, we first need to determine the gain in terms of the S-parameters of the transistor. Because there are forward and reverse waves at the input and output ports, unlike in circuit theory, there several different power measures that can be used to define gain:

- $P_{\mathrm{in}} =$ power delivered to network
- $P_{\mathrm{avs}} =$power available from source
- $P_L =$ power delivered to load
- $P_{\mathrm{avn}} =$ power available from network

Available power is the maximum power that can be supplied with over all possible load impedances. The actual power delivered may be smaller than the available power due to reflections and is less than or equal to the available power.

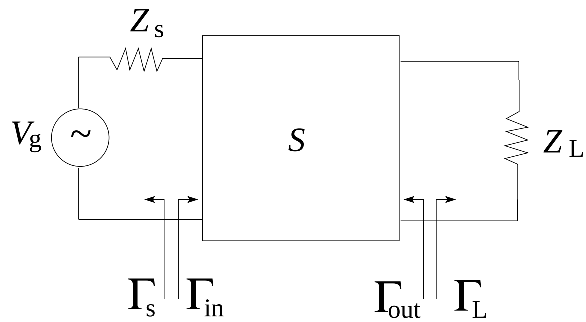

Figure 3.1: A microwave amplifier connected to a source at the input (port 1) and a load impedance at the output (port 2).

We first need to find the reflection coefficients looking into ports 1 and 2, as shown in Fig. 3.1. The source and load reflection coefficients are

$$\Gamma_s = \frac{Z_s - Z_0}{Z_s + Z_0} \qquad \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{3.1}$$

Given these definitions, we can write the amplitudes b_1 and b_2 of the waves exiting ports 1 and 2 as

$$b_1 = S_{11}a_1 + S_{12}a_2 = S_{11}a_1 + S_{12}\Gamma_L b_2 \tag{3.2}$$

$$b_2 = S_{21}a_1 + S_{22}a_2 = S_{21}a_1 + S_{22}\underbrace{\Gamma_L b_2}_{a_2} \tag{3.3}$$

Solving the second equation for b_2 gives

$$b_2 = \frac{S_{21}}{1 - S_{22}\Gamma_L}a_1 \tag{3.4}$$

Using this expression in the first equation gives

$$b_1 = S_{11}a_1 + S_{12}\Gamma_L\frac{S_{21}}{1 - S_{22}\Gamma_L}a_1 \tag{3.5}$$

The ratio b_1/a_1 gives the reflection coefficient looking into the input port:

$$\Gamma_{\text{in}} = \frac{b_1}{a_1} = S_{11} + \frac{S_{12}\Gamma_L S_{21}}{1 - S_{22}\Gamma_L} = \frac{Z_{\text{in}} - Z_0}{Z_{\text{in}} + Z_0} \tag{3.6}$$

Repeating this procedure for a source at port 2, we find that

$$\Gamma_{\text{out}} = S_{22} + \frac{S_{12}\Gamma_s S_{21}}{1 - S_{11}\Gamma_s} \tag{3.7}$$

This expression can be understood in an intuitive way. The first term accounts for reflections at port 2, and would be the entire reflection coefficient if port 1 were matched. In the second term, S_{12} takes the signal from port 2 to port 1, Γ_s reflects the signal from the source impedance, S_{21} returns the signal to port 2, and the denominator takes into account multiple reflections between the ports.

We can compute the various power quantities in terms of these reflection coefficients, using

$$\begin{aligned} P_{\text{in}} &= |a_1|^2/2 - |b_1|^2/2 \\ &= |a_1|^2/2\left(1 - |\Gamma_{in}|^2\right) \end{aligned} \tag{3.8}$$

$$\begin{aligned} P_L &= |b_2|^2/2 - |a_2|^2/2 \\ &= |b_2|^2/2\left(1 - |\Gamma_L|^2\right) \end{aligned} \tag{3.9}$$

Now, we need to find a_1 and b_2. We can get the voltage at port 1 using a voltage divider,

$$V_1 = V_s \frac{Z_{\text{in}}}{Z_s + Z_{\text{in}}} \tag{3.10}$$

$$= \sqrt{Z_0}(a_1 + b_1) \quad \text{by the definition of generalized S-parameters} \tag{3.11}$$

$$= \sqrt{Z_0} a_1 (1 + \Gamma_{\text{in}}) \tag{3.12}$$

Substituting the definition of Z_{in} in terms of Γ_{in} into this expression gives

$$V_s \frac{(1+\Gamma_{\text{in}})Z_0}{Z_s(1-\Gamma_{\text{in}}) + Z_0(1+\Gamma_{\text{in}})} = \sqrt{Z_0} a_1 (1+\Gamma_{\text{in}}) \tag{3.13}$$

Solving for a_1 and putting the expression in terms of reflection coefficients gives

$$\begin{aligned} a_1 &= \frac{Z_0}{Z_s(1-\Gamma_{\text{in}}) + Z_0(1+\Gamma_{\text{in}})} \frac{V_s}{\sqrt{Z_0}} \\ &= \frac{Z_0}{Z_0 \frac{1+\Gamma_s}{1-\Gamma_s}(1-\Gamma_{\text{in}}) + Z_0(1+\Gamma_{\text{in}})} \frac{V_s}{\sqrt{Z_0}} \\ &= \frac{1-\Gamma_s}{(1+\Gamma_s)(1-\Gamma_{\text{in}}) + (1-\Gamma_s)(1+\Gamma_{\text{in}})} \frac{V_s}{\sqrt{Z_0}} \\ &= \frac{1}{2} \frac{1-\Gamma_s}{1-\Gamma_{\text{in}}\Gamma_s} \frac{V_s}{\sqrt{Z_0}} \end{aligned} \tag{3.14}$$

If we put this into the expression for P_{in}, we find that

$$\begin{aligned} P_{\text{in}} &= \frac{1}{2} \left| \frac{1}{2} \frac{1-\Gamma_s}{1-\Gamma_{\text{in}}\Gamma_s} \frac{V_s}{\sqrt{Z_0}} \right|^2 (1 - |\Gamma_{\text{in}}|^2) \\ &= \frac{|V_s|^2}{8Z_0} \frac{|1-\Gamma_s|^2}{|1-\Gamma_{\text{in}}\Gamma_s|^2} (1 - |\Gamma_{\text{in}}|^2) \end{aligned} \tag{3.15}$$

Using (3.4) together with (3.14) gives the amplitude of the wave propagating out of port 2,

$$b_2 = \frac{S_{21}}{1 - S_{22}\Gamma_L} a_1 = \frac{V_s}{2\sqrt{Z_0}} \frac{S_{21}(1-\Gamma_s)}{(1-\Gamma_{\text{in}}\Gamma_s)(1-S_{22}\Gamma_L)} \tag{3.16}$$

This can be used to obtain the power delivered to the load,

$$P_L = \frac{|V_s|^2}{8Z_0} |S_{21}|^2 \frac{(1-|\Gamma_L|^2)|1-\Gamma_s|^2}{|1-S_{22}\Gamma_L|^2 |1-\Gamma_s\Gamma_{\text{in}}|^2} \tag{3.17}$$

Power Gain

Using these results, the power gain of the amplifier is

$$\begin{aligned} G_P = \frac{P_L}{P_{\text{in}}} &= |S_{21}|^2 \frac{(1-|\Gamma_L|^2)|1-\Gamma_s|^2}{|1-S_{22}\Gamma_L|^2|1-\Gamma_s\Gamma_{\text{in}}|^2} \frac{|1-\Gamma_s\Gamma_{\text{in}}|^2}{|1-\Gamma_s|^2(1-|\Gamma_{\text{in}}|^2)} \\ &= |S_{21}|^2 \frac{1-|\Gamma_L|^2}{|1-S_{22}\Gamma_L|^2(1-|\Gamma_{in}|^2)} \end{aligned} \tag{3.18}$$

This is the gain of the amplifier in terms of power delivered to the load relative to power coming into the amplifier input port.

In this expression, if the load reflection coefficient is one, then the gain vanishes, as expected. If the load reflection coefficient is zero, then

$$G_P = |S_{21}|^2 \frac{1}{1-|\Gamma_{\text{in}}|^2} = |S_{21}|^2 \frac{1}{1-|S_{11}|^2} \tag{3.19}$$

Note, however, that $\Gamma_L = 0$ does not mean the power gain is maximized, since it simply means $Z_L = Z_0$ whereas maximum power gain occurs for a conjugate match condition (we will see this more clearly later). More generally, the power gain is equal to $|S_{21}|^2$ scaled by a factor which takes into account reflections at the load (which means G_P can be larger than $|S_{21}|^2$).

Transducer Power Gain

The problem with power gain as defined in Eq. (3.19) is that it does not consider the match between the source and the input impedance to the network. In some cases, therefore, a more meaningful measure of gain is the power dissipated by the load relative to the maximum power that the source can supply. This is the transducer power gain,

$$G_T = \frac{P_L}{P_{\text{avs}}} \tag{3.20}$$

If we consider a source with impedance Z_s driving a line with input impedance Z_{in}, the maximum power transfer occurs when Z_{in} is the complex conjugate of Z_s, so that

$$Z_{\text{in}} = Z_s^* \tag{3.21}$$

This can be proved by taking the derivative of the power delivered to the line with respect to the real and imaginary parts of Z_{in} and setting the derivatives to zero. The imaginary parts of Z_{in} and Z_s are equal in magnitude and opposite in sign, which is what happens with the impedances of the inductor and capacitor in an LCR circuit at resonance. This is called a **conjugate match**.

With a conjugate match to the source impedance, the power available from the source is

$$\begin{aligned} P_{\text{avs}} &= P_{\text{in}}\big|_{\Gamma_{\text{in}}=\Gamma_s^*} \quad \text{(conjugate match)} \\ &= \frac{|V_s|^2}{8Z_0} \frac{|1-\Gamma_s|^2}{|1-\underbrace{\Gamma_s^*}_{\text{cm}}\Gamma_s|^2}(1-|\Gamma_s|^2) \\ &= \frac{|V_s|^2}{8Z_0} \frac{|1-\Gamma_s|^2}{1-|\Gamma_s|^2} \end{aligned} \tag{3.22}$$

The transducer gain is then

$$\begin{aligned} G_T &= |S_{21}|^2 \frac{(1-|\Gamma_L|^2)|1-\Gamma_s|^2}{|1-S_{22}\Gamma_L|^2|1-\Gamma_s\Gamma_{\rm in}|^2} \frac{1-|\Gamma_s|^2}{|1-\Gamma_s|^2} \\ &= |S_{21}|^2 \frac{(1-|\Gamma_s|^2)(1-|\Gamma_L|^2)}{|1-S_{22}\Gamma_L|^2|1-\Gamma_s \underbrace{\Gamma_{\rm in}}_{\neq \text{cm}}|^2} \end{aligned} \tag{3.23}$$

Notice that the input reflection coefficient in the power delivered to the load does not assume a conjugate match at the input. This is because transducer gain does not mean that we have actually conjugate matched to the input impedance. Instead, we just want to compute the gain relative to the input power we would have if the source were conjugate matched.

Available Power Gain

Another difficulty with power gain in Eq. (3.19) is that if the load is not well matched to the network output impedance, then the power gain is small, even though the amplifier is capable of delivering more power to a better matched load. If we want to characterize the amplifier independently of the load impedance, we can define a measure of gain in terms of the power delivered to a conjugate matched load. This is the available power gain,

$$G_A = \frac{P_{\rm avn}}{P_{\rm avs}} \tag{3.24}$$

which is the gain if both the source and load were conjugate matched. The power available from the amplifier network at port 2 is

$$\begin{aligned} P_{\rm avn} &= P_L|_{\Gamma_L=\Gamma_{\rm out}^*} \\ &= \frac{|V_s|^2}{8Z_0}|S_{21}|^2 \frac{(1-|\Gamma_{\rm out}|^2)|1-\Gamma_s|^2}{|1-S_{22}\Gamma_{\rm out}^*|^2|1-\Gamma_s\Gamma_{\rm in}|^2} \end{aligned} \tag{3.25}$$

The available gain is

$$G_A = |S_{21}|^2 \frac{(1-|\Gamma_{\rm out}|^2)(1-|\Gamma_s|^2)}{|1-S_{22}\Gamma_{\rm out}^*|^2|1-\Gamma_s\Gamma_{\rm in}|^2} \tag{3.26}$$

In order to simplify the expression for available power gain by eliminating $\Gamma_{\rm in}$, we use

$$\begin{aligned} 1-\Gamma_s\Gamma_{\rm in} &= 1-\Gamma_s\left(S_{11}+\frac{S_{12}\Gamma_L S_{21}}{1-S_{22}\Gamma_L}\right) \\ &= \frac{1-S_{22}\Gamma_L-S_{11}\Gamma_s+S_{11}S_{22}\Gamma_s\Gamma_L-S_{12}S_{21}\Gamma_s\Gamma_L}{1-S_{22}\Gamma_L} \\ &= \frac{1-S_{11}\Gamma_s}{1-S_{22}\Gamma_L}\left[1-\Gamma_L\underbrace{\left(S_{22}+\frac{S_{12}\Gamma_s S_{21}}{1-S_{11}\Gamma_s}\right)}_{\Gamma_{\rm out}}\right] \\ &= \frac{1-S_{11}\Gamma_s}{1-S_{22}\Gamma_L}(1-\Gamma_L\Gamma_{\rm out}) \end{aligned} \tag{3.27}$$

With this result, the available gain becomes

$$G_A = |S_{21}|^2 \frac{1-|\Gamma_s|^2}{|1-S_{11}\Gamma_s|^2(1-|\Gamma_{\rm out}|^2)} \tag{3.28}$$

Special Cases

1. Both source and load are matched ($\Gamma_s = \Gamma_L = 0$). In this case,

$$G_T = |S_{21}|^2 \tag{3.29}$$

   It is important to be aware that if we instead choose a conjugate match at the source and load ($\Gamma_s = \Gamma_{\text{in}}^*$, $\Gamma_L = \Gamma_{\text{out}}^*$), the gain can be larger than $|S_{21}|^2$.

2. Unilateral device ($S_{12} = 0$ or very small). In this case, $\Gamma_{\text{in}} = S_{11}$ and $\Gamma_{\text{out}} = S_{22}$, and the transducer gain becomes

$$\begin{aligned} G_{TU} &= |S_{21}|^2 \frac{(1-|\Gamma_s|^2)(1-|\Gamma_L|^2)}{|1-S_{22}\Gamma_L|^2|1-\Gamma_s S_{11}|^2} \\ &= \underbrace{\frac{1-|\Gamma_s|^2}{|1-\Gamma_s S_{11}|^2}}_{\substack{\text{source} \\ G_s}} \underbrace{|S_{21}|^2}_{\substack{\text{device} \\ G_o}} \underbrace{\frac{1-|\Gamma_L|^2}{|1-S_{22}\Gamma_L|^2}}_{\substack{\text{load} \\ G_L}} \end{aligned} \tag{3.30}$$

   which can be broken up into a product of source, device, and load gain factors. Again, it is possible for G_s and G_L to be greater than one, depending on the source and load matches.

   Using the expression for transducer gain for a bilateral device, it can be shown that the error made in assuming that a transistor is unilateral is bounded by

$$\frac{1}{(1+U)^2} \leq \frac{G_T}{G_{TU}} < \frac{1}{(1-U)^2} \tag{3.31}$$

   where

$$U = \frac{S_{12}S_{21}S_{11}S_{22}}{(1-|S_{11}|^2)(1-|S_{22}|^2)} \tag{3.32}$$

   is the unilateral figure of merit. The smaller U, the closer the device is to unilateral.

Summary

For a two-port amplifier, we define three gain quantities:

$$G_P = \frac{P_L}{P_{\text{in}}} \quad \text{(Power Gain)} \tag{3.33}$$

$$G_T = \frac{P_L}{P_{\text{avs}}} \quad \text{(Transducer Gain)} \tag{3.34}$$

$$G_A = \frac{P_{\text{avn}}}{P_{\text{avs}}} \quad \text{(Available Power Gain)} \tag{3.35}$$

In developing amplifier design procedures, we will use whichever type of gain is most convenient for a given problem. For unilateral amplifier design ($S_{12} = 0$), we will use transducer gain, and for bilateral design ($S_{12} \neq 0$) we will use power gain.