

Chapter 1

Transmission Lines and S-Parameters

1.1 Review of Transmission Lines

In our study of microwave circuits, we will be extensively using our knowledge of transmission lines. We therefore need to review a few concepts from transmission line theory. We will only need the sinusoidal steady state, so line voltages and currents will be in phasor form. The phasor voltage $V(z)$ and current $I(z)$ are defined in terms of the time-dependent voltage according to

$$v(z, t) = \text{Re} \{ V(z) e^{j\omega t} \} \quad (1.1)$$

$$i(z, t) = \text{Re} \{ I(z) e^{j\omega t} \} \quad (1.2)$$

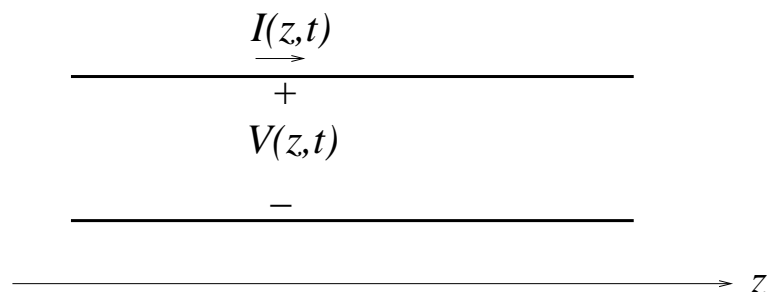


Figure 1.1: A transmission line. The two wires are only a schematic representation, because many different types of structures can be modeled as transmission lines.

1. Differential (Telegrapher's) Equations:

The governing equations for the line current and voltage at a point z on a lossless line are

$$\frac{\partial v(z, t)}{\partial z} = -L \frac{\partial i(z, t)}{\partial t} \quad (1.3)$$

$$\frac{\partial i(z, t)}{\partial z} = -C \frac{\partial v(z, t)}{\partial t} \quad (1.4)$$

where L (H/m) and C (F/m) are the distributed inductance and capacitance, respectively. In the sinusoidal steady state,

$$\frac{dV(z)}{dz} = -j\omega LI(z) \quad (1.5)$$

$$\frac{dI(z)}{dz} = -j\omega CV(z) \quad (1.6)$$

Substituting the second equation into the first gives

$$\frac{d^2V(z)}{dz^2} = -j\omega L \frac{dI(z)}{dz} = -\omega^2 LCV(z) \quad (1.7)$$

which leads to the Helmholtz equation,

$$\frac{d^2V(z)}{dz^2} + \underbrace{\omega^2 LC}_{\beta} V(z) = 0 \quad (1.8)$$

where $\beta = \omega\sqrt{LC}$ is the wavenumber of the sinusoidal waves.

The general solution to this differential equation is

$$V(z) = Ae^{-j\beta z} + Be^{+j\beta z} \quad (1.9)$$

where A and B are unknown coefficients. The corresponding current is

$$\begin{aligned} I(z) &= -\frac{1}{j\omega L} \frac{dV(z)}{dz} = \frac{\beta}{\omega L} [Ae^{-j\beta z} - Be^{j\beta z}] \\ &= \frac{1}{\sqrt{L/C}} [Ae^{-j\beta z} - Be^{j\beta z}] \end{aligned} \quad (1.10)$$

We define

$$Z_o = \sqrt{\frac{L}{C}} \quad (1.11)$$

as the *characteristic impedance* of the transmission line, because it gives the ratio between the voltage and current waves at a point on the line. Note that this quantity is nonzero even though the line is lossless. What is the difference between characteristic impedance of a transmission line and the impedance of a circuit element?

Using the characteristic impedance, we can write the current on the line as

$$I(z) = \frac{A}{Z_o} e^{-j\beta z} - \frac{B}{Z_o} e^{j\beta z} \quad (1.12)$$

In the time domain,

$$v(z, t) = \text{Re} \left\{ Ae^{j(\omega t - \beta z)} + Be^{j(\omega t + \beta z)} \right\} = A \cos(\omega t - \beta z) + B \cos(\omega t + \beta z) \quad (1.13)$$

so that the A term represents a forward traveling ($+z$) wave and the B term is the reverse wave.

2. **Wavelength:** This is the distance between two peaks of the forward or reverse wave:

$$\beta z = \beta \lambda = 2\pi \quad \rightarrow \quad \lambda = \frac{2\pi}{\beta} \quad (1.14)$$

3. **Phase Velocity:** In order to stay on a peak of the wave, the argument of one of the cosine functions in (1.13) must be constant. From this idea, we can determine the phase velocity of the wave:

$$\omega t - \beta z = \phi = \text{constant} \tag{1.15}$$

$$z = \frac{\omega t - \phi}{\beta} \tag{1.16}$$

$$v_p = \frac{dz}{dt} = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} \tag{1.17}$$

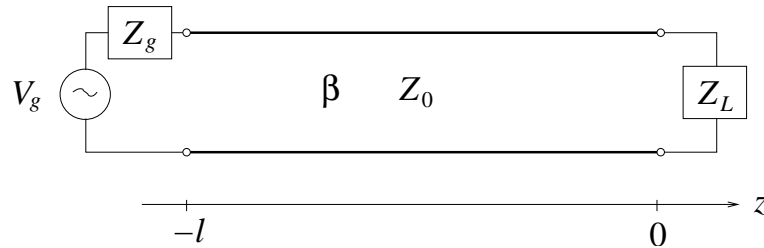


Figure 1.2: A transmission line with generator at $z = -l$ and load at $z = 0$.

4. **Reflection Coefficient:** Consider a line of characteristic impedance Z_o and length ℓ terminated with a load impedance of Z_L , as shown in Fig. 1.2. Let $z = 0$ be defined at the load, such that the input is at $z = -\ell$. From the general solutions derived above, the voltage and current on the line are

$$V(z) = Ae^{-j\beta z} + Be^{+j\beta z} \tag{1.18}$$

$$I(z) = \frac{A}{Z_o}e^{-j\beta z} - \frac{B}{Z_o}e^{j\beta z} \tag{1.19}$$

The coefficients A and B are determined by the boundary conditions at the load and generator ends of the transmission line. At the load end, the current and voltage on the line must be equal to the voltage across the load and the current through the load, which are related by Ohm's law, so that

$$Z_L = \frac{V(0)}{I(0)} = Z_o \frac{A + B}{A - B} \tag{1.20}$$

At the generator end, Kirchhoff's voltage law gives

$$\begin{aligned} V_g &= V(-\ell) + Z_g I(-\ell) \\ &= Ae^{j\beta\ell} + Be^{-j\beta\ell} + Z_g \left[\frac{A}{Z_o}e^{j\beta\ell} - \frac{B}{Z_o}e^{-j\beta\ell} \right] \end{aligned} \tag{1.21}$$

Equations (1.20) and (1.21) provide two simultaneous equations that can be solved for the two unknowns in the problem, A and B . Once we have these two coefficients, we know the current and voltage everywhere on the line, and the problem is solved, because we can get any derived quantity that we want from the current and voltage.

We could solve Eqs. (1.20) and (1.21) using any method for solving systems of equations. It is convenient to solve these equations in a way that gives values for two very useful quantities, the load reflection coefficient at $z = 0$ and the input impedance at $z = -\ell$. To do this, we first solve the load equation (1.20) for B/A , which is the load reflection coefficient:

$$\Gamma_o = \frac{B}{A} = \frac{Z_L - Z_o}{Z_L + Z_o} \tag{1.22}$$

We can also define a generalized reflection coefficient at other points on the transmission line as

$$\Gamma(z) = \frac{V^-(z)}{V^+(z)} = \frac{Be^{+j\beta z}}{Ae^{-j\beta z}} = \Gamma_o e^{j2\beta z} \quad (1.23)$$

5. **Input Impedance:** At the generator end of the line ($z = -\ell$),

$$Z_{\text{in}} = \frac{V(-\ell)}{I(-\ell)} = Z_o \frac{Ae^{j\beta\ell} + Be^{-j\beta\ell}}{Ae^{j\beta\ell} - Be^{-j\beta\ell}} = Z_o \frac{1 + \Gamma_o e^{-j2\beta\ell}}{1 - \Gamma_o e^{-j2\beta\ell}} \quad (1.24)$$

Using (1.22) and (1.24),

$$Z_{\text{in}} = Z_o \frac{(Z_L + Z_o)e^{j\beta\ell} + (Z_L - Z_o)e^{-j\beta\ell}}{(Z_L + Z_o)e^{j\beta\ell} - (Z_L - Z_o)e^{-j\beta\ell}} = Z_o \frac{Z_L + jZ_o \tan \beta\ell}{Z_o + jZ_L \tan \beta\ell} \quad (1.25)$$

This can be used with the generator voltage and impedance to solve for $V(-\ell)$. Using $V(-\ell)$ together with Eq. (1.18) written in the form

$$V(-\ell) = A(e^{j\beta\ell} + \Gamma_o e^{-j\beta\ell}) \quad (1.26)$$

allows us to find A , which completes the solution of the transmission line problem in Fig. 1.2.

6. **VSWR:** The voltage standing wave ratio is the ratio of the maximum magnitude of the voltage along the line to the minimum:

$$V(z) = Ae^{-j\beta z} [1 + \Gamma_o e^{j2\beta z}] \quad (1.27)$$

$$|V(z)|_{\text{max}} = |A| [1 + |\Gamma_o|] \quad (1.28)$$

$$|V(z)|_{\text{min}} = |A| [1 - |\Gamma_o|] \quad (1.29)$$

$$\text{VSWR} = \frac{1 + |\Gamma_o|}{1 - |\Gamma_o|} \quad (1.30)$$

7. **Matched Line:**

$$Z_L = Z_o \quad \Gamma_o = 0$$

$$Z_{\text{in}}(-\ell) = Z_o \quad \text{VSWR} = 1$$

8. **Short-Circuited Line:**

$$Z_{\text{in}}(-\ell) = Z_o \frac{jZ_o \tan \beta\ell}{Z_o} = jZ_o \tan \beta\ell \quad (1.31)$$

9. **Open-Circuited Line:**

$$Z_{\text{in}}(-\ell) = \frac{Z_o}{j \tan \beta\ell} = -jZ_o \cot \beta\ell \quad (1.32)$$

10. **Quarter-Wave Line:**

$$Z_{\text{in}}(-\ell = -\lambda/4) = Z_o \frac{Z_L + jZ_o \tan \pi/2}{Z_o + jZ_L \tan \pi/2} = \frac{Z_o^2}{Z_L} \quad (1.33)$$

1.2 Microwave Networks and S-Parameters

Previously in our study of transmission line theory, we have focused mainly on the transmission line itself. Now, we would like to view transmission lines merely as connections between devices, and shift our emphasis to the properties of the devices themselves.

A microwave network is a device or structure to which one or more transmission lines are connected. Each transmission line connection to the device is a port. An N -port network is shown in Fig. 1.3. Power splitters, lumped elements, amplifiers, antennas, a section of transmission line, and many other structures can be modeled as networks.

As with a circuit device, a network can be characterized in terms of the voltage/current relationship at each transmission line port. For a simple parallel wire transmission line, defining the voltage at a point on the line is straightforward. But for waveguides, voltages are more difficult to define for non-TEM modes. Even for a TEM mode, voltages and currents are difficult to measure directly at microwave frequencies, because of the rapid oscillation with time. Consider, for example, how fast a digital sampling oscilloscope would have to operate to accomplish this measurement.

Instead of using voltages and currents to characterize the signals on the transmission line ports, we use a representation that is based on the relative amplitudes of incident, reflected, and transmitted waves for a structure. Regardless of what the network may be physically, on each transmission line port there can only be a forward wave (incident) and a reverse wave (reflected or transmitted by the network) with some steady state amplitude. This means that instead of characterizing the signal on a transmission line using $V(z)$ or $I(z)$, we use the ratios between the forward and reverse waves on the transmission lines. For a network connected at each port by transmission lines to other devices, each transmission line will have a given value for this ratio. If the other devices are all matched and a unit amplitude forward wave is excited as an input into one of the ports, then the values of the reverse waves coming out the ports are completely determined by the device and can be used to characterize the microwave properties of the device. These values are called the *scattering matrix* or *S-matrix*. We will generally use the term *S-parameters* to describe the elements of the S-matrix.

The S-matrix is defined to relate the incident and reflected wave amplitudes at the ports of the network:

$$\begin{bmatrix} V_1^- \\ V_2^- \\ \vdots \\ V_N^- \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1N} \\ S_{21} & S_{22} & \cdots & S_{2N} \\ \vdots & & & \vdots \\ S_{N1} & S_{N2} & \cdots & S_{NN} \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \\ \vdots \\ V_N^+ \end{bmatrix} \quad (1.34)$$

or

$$[V^-] = [S] [V^+] \quad (1.35)$$

From this expression, it can be seen that in general, the output wave at a given port is a function of the input waves at every port. If there is no input wave on any port except for the j th port, then all of the elements of the vector V^+ are zero except for V_j^+ , and the S-parameters can be expressed as

$$S_{ij} = \left. \frac{V_i^-}{V_j^+} \right|_{V_k^+ = 0 \text{ for } k \neq j} \quad (1.36)$$

This formula shows that an S-parameter is similar to a generalized reflection coefficient, since it is defined the same way as $\Gamma(z)$ in Eq. (1.23). The only difference between (1.36) and (1.23) is that the forward

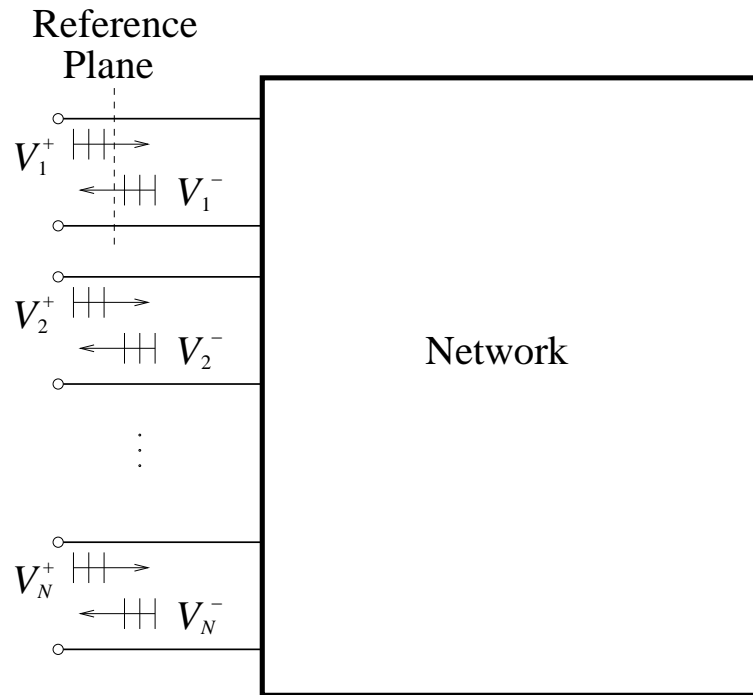


Figure 1.3: N -port network. Each port is a transmission line, leading to a device or structure that connects the ports. At a fixed reference plane for each port, the voltage amplitudes of the forward and reverse traveling waves are V_n^+ and V_n^- , respectively.

wave is on the j th port and the reverse wave is on the i th port. Equation (1.36) shows how to measure the S-parameters S_{ij} : inject a known signal into port j and measure the signal at port i with all other ports terminated with a matched load.

1.2.1 Reference Planes

The S-parameter S_{ij} is a complex number that can be represented by a magnitude and phase. The phase of the ratio in Eq. (1.36) depends on the location along the transmission line at which we measure the S-parameter. The location at which the measurement is made is called the reference plane.

We can derive an expression for how the S-matrix changes if the reference planes at each port are changed. For a N -port network, $[V^-] = [S][V^+]$. If we change the reference position of the n th port a distance ℓ_n away from the device, the new input and output waves are related to the new S-matrix by $[V^{-'}] = [S'][V^{+'}]$. From transmission line theory,

$$\begin{aligned}
 V_n^{+'} &= V_n^+ e^{j\beta_n \ell_n} = V_n^+ e^{j\theta_n} & \rightarrow & & V_n^+ &= V_n^{+'} e^{-j\theta_n} \\
 V_n^{-'} &= V_n^- e^{-j\beta_n \ell_n} = V_n^- e^{-j\theta_n} & \rightarrow & & V_n^- &= V_n^{-'} e^{j\theta_n}
 \end{aligned} \tag{1.37}$$

Placing these equations into matrix form gives

$$\begin{bmatrix} e^{j\theta_1} & 0 & \dots & 0 \\ 0 & e^{j\theta_2} & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & e^{j\theta_N} \end{bmatrix} [V^{-'}] = [S] \begin{bmatrix} e^{-j\theta_1} & 0 & \dots & 0 \\ 0 & e^{-j\theta_2} & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & e^{-j\theta_N} \end{bmatrix} [V^{+'}] \quad (1.38)$$

or

$$[V^{-'}] = \underbrace{[\Theta][S][\Theta]}_{[S']} [V^{+'}] = [S'] [V^{+'}] \quad (1.39)$$

where Θ is the matrix of phase shifts in Eq. (1.38). This result shows how the scattering matrices for the same network but with a different choice of reference planes are related.

1.2.2 Generalized S-Parameters

So far we have assumed that the characteristic impedances of all of the transmission line port are the same. This is often the case. But in many practical situations, the impedances are different. In this case, for convenience we redefine the S-parameters so that they are related in a simple way to the power flowing through each of the ports.

Recall from transmission line analysis that the average (real) power associated with a voltage wave is given as $|V|^2/2Z_o$. Therefore, we can write for a two-port device where the reference impedance is different for the two ports (and the device is matched)

$$P_{in} = \frac{|V_1^+|^2}{2Z_{o1}} \quad (1.40)$$

$$P_{out} = \frac{|V_2^-|^2}{2Z_{o2}} \quad (1.41)$$

so that

$$\frac{P_{out}}{P_{in}} = \frac{|V_2^-|^2 Z_{o1}}{|V_1^+|^2 Z_{o2}} \quad (1.42)$$

If we define generalized S-parameters by

$$a_n = \frac{V_n^+}{\sqrt{Z_{on}}} \quad b_n = \frac{V_n^-}{\sqrt{Z_{on}}} \quad S_{ij} = \left. \frac{b_i}{a_j} \right|_{a_k=0 \text{ for } k \neq j} \quad (1.43)$$

then

$$P_{in} = \frac{1}{2} |a_1|^2 \quad (1.44)$$

$$P_{out} = \frac{1}{2} |b_2|^2 \quad (1.45)$$

The ratio of input and output power is

$$\frac{P_{out}}{P_{in}} = \frac{|b_2|^2}{|a_1|^2} = |S_{21}|^2 \quad (1.46)$$

which is more convenient than (1.42). Notice that generalized S-parameters reduce to the same definition as in (1.34) if the impedances at each port are the same.

1.2.3 Lossless Networks

For lossless networks, no real power is delivered to the network. The time average power delivered to each port is

$$\begin{aligned}
 P_{av} &= \frac{1}{2} \text{Re} \{ [V]^T [I]^* \} \\
 &= \frac{1}{2} \text{Re} \left\{ ([a]^T + [b]^T) [\sqrt{Z_o}] [\sqrt{Z_o}]^{-1} ([a]^* - [b]^*) \right\} \\
 &= \frac{1}{2} \text{Re} \left\{ ([a]^T [a]^* - [a]^T [b]^* + [b]^T [a]^* - [b]^T [b]^*) \right\}
 \end{aligned} \tag{1.47}$$

where $[\sqrt{Z_o}]$ is a diagonal matrix with n th diagonal entry $\sqrt{Z_{on}}$ and the superscript T represents the matrix transpose operation. The middle two terms of this expression give an imaginary result. The other two terms are purely real. But since P_{av} is the average power delivered to the network, the real part must be zero:

$$P_{av} = \frac{1}{2} ([a]^T [a]^* - [b]^T [b]^*) = 0 \tag{1.48}$$

The first and second terms represent the power entering and exiting the ports, respectively. This equation implies that

$$\begin{aligned}
 [a]^T [a]^* = [b]^T [b]^* &= ([S][a])^T ([S][a])^* \\
 &= [a]^T [S]^T [S]^* [a]^*
 \end{aligned} \tag{1.49}$$

which in turn implies that

$$[S]^T [S]^* = I = \text{Identity Matrix} \tag{1.50}$$

A matrix that satisfies this relationship is *unitary*. We can also write this condition as

$$\sum_{k=1}^N S_{ik}^T S_{kj}^* = \delta_{ij} \tag{1.51}$$

or

$$\sum_{k=1}^N S_{ki} S_{kj}^* = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{1.52}$$

Equations (1.50)-(1.52) all state the same condition, which if satisfied by the scattering matrix means that the network must be lossless.

1.2.4 Reciprocity

A network is reciprocal if

$$[S] = [S^T] \tag{1.53}$$

where the superscript denotes the matrix transpose operation. Consider the S-parameters of an isolator, which only allows a one-way flow of energy through a two-port network. Would the matrix satisfy (1.53)?

Reciprocity should not be confused with network symmetry. For a 2-port device, network symmetry means that $S_{11} = S_{22}$, so that port 1 behaves like port 2 if a signal is input to one of the ports and the other port is terminated with a matched load. (as well as the $S_{12} = S_{21}$ imposed by (1.53) if the network is reciprocal).

Chapter 2

Passive Structures

2.1 Lumped Element Matching

Matching is a key part of any RF/Microwave circuit design. The basic concept of matching is to couple energy from a transmission line into a load with as little reflection as possible. But in practice, matching is used for much more than simply minimizing reflection coefficients. Using appropriate matching techniques we can control gain, noise figure, and stability for a device.

There are a variety of techniques available for matching. We will first consider the simple but very useful case of using two lumped elements to accomplish a match at a single frequency. There are eight potential topologies available of this form (not all of which will be suitable for a given problem). Two possible matching configurations are shown in Fig. 2.1.

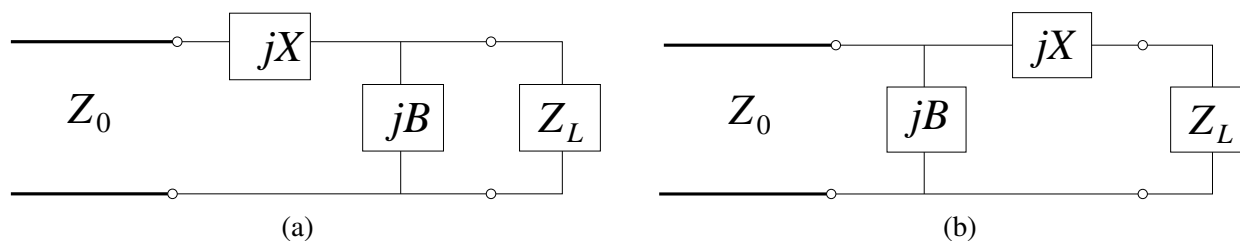


Figure 2.1: Matching network topologies. (a) Series reactance followed by a shunt susceptance. (b) Shunt susceptance followed by a series reactance.

2.1.1 Analytic Solution

We begin by formulating an analytic solution for this matching problem. Consider the case of a series reactance followed by a shunt susceptance. Let $Z_L = R_L + jX_L$ with admittance $Y_L = G_L + jB_L$. In order to match the input to Z_0 , we must have

$$Z_o = jX + \frac{1}{jB + G_L + jB_L} \quad (2.1)$$

If we cross multiply and separate into real and imaginary parts, we obtain the two equations

$$1 - (B_L + B)X = G_L Z_o \quad (2.2)$$

$$(B_L + B)Z_o = G_L X \quad (2.3)$$

If we solve the second equation for X and put this into the first equation, we obtain the solutions

$$B = \pm \sqrt{G_L(Y_o - G_L)} - B_L \quad (2.4)$$

$$X = \pm \frac{\sqrt{(Y_o - G_L)/G_L}}{Y_o} \quad (2.5)$$

where if you choose the top (bottom) sign for the first equation you must use the top (bottom) sign for the second. So, this topology provides two possible solutions depending on the sign used. Note that in order for the term in the radical to remain positive, we must have $Y_o > G_L$.

If we repeat the analysis for the topology in Fig. 2.1b, a shunt susceptance followed by a series reactance, we obtain the solutions

$$X = \pm \sqrt{R_L(Z_o - R_L)} - X_L \quad (2.6)$$

$$B = \pm \frac{\sqrt{(Z_o - R_L)/R_L}}{Z_o} \quad (2.7)$$

where if you choose the top (bottom) sign for the first equation, you must use the top (bottom) sign for the second equation. In this case, we must have $Z_o > R_L$.

Note that in these derivations, we are matching a complex load to a real impedance. We could also match a complex load to a complex impedance using these networks.

2.1.2 Smith Chart Solution

It is perhaps most instructive to demonstrate lumped-element matching on the Smith Chart using an example. Suppose we want to match $Z_L = 25\Omega$ to a 50Ω line at 1 GHz..

1. Normalize Z_L :

$$z_L = \frac{Z_L}{Z_o} = 0.5$$

Pick a topology: Since $R_L < Z_o$, we must use the topology with a shunt susceptance followed by a series reactance.

2. Work from the load to the source impedance. We want to use the series reactance to get a normalized admittance of the form $1 + jx$, so we can use the shunt susceptance to cancel the jx part and get a match (normalized impedance = 1). We do this by adding enough reactance to move to the reflected $1 + jx$ circle, so from the Smith chart, there are two points we can use, $x = 0.5$, and $x = -0.5$. We then reflect to an admittance, so we can add the shunt susceptance.
3. Now, add enough susceptance to move to the center of the Smith Chart: $b = 1.0$, or $b = -1.0$ for the other solution.

4. Convert the values to components:

$$X = Z_o x = \begin{cases} 2\pi f L_s & X > 0 \\ -\frac{1}{2\pi f C_s} & X < 0 \end{cases}$$

$$B = Y_o b = \frac{b}{Z_o} = \begin{cases} 2\pi f C_p & B > 0 \\ -\frac{1}{2\pi f L_p} & B < 0 \end{cases}$$

For this example, we get

$$jX = +j25 = j\omega L \Rightarrow L \simeq 4 \text{ nH}$$

$$jB = +j/50 = j\omega C \Rightarrow C \simeq 3.2 \text{ pF}$$

or $C \simeq 6.4 \text{ pF}$, $L \simeq 8 \text{ nH}$ for the other solution.

5. At higher frequencies, you might have to realize these components using transmission line stubs instead of lumped elements. In this case:

Open-Circuit Stubs

$$x = -\frac{Z_o \cot \beta \ell}{Z_o} = -\cot \beta \ell$$

$$b = \frac{Y_o \tan \beta \ell}{Y_o} = \tan \beta \ell$$

Short-Circuit Stubs

$$x = \frac{Z_o \tan \beta \ell}{Z_o} = \tan \beta \ell$$

$$b = -\frac{Y_o \cot \beta \ell}{Y_o} = -\cot \beta \ell$$

You can then solve for the appropriate length.

Note that we can very easily use combinations of transmission lines and lumped elements when we are using the Smith Chart to match.

2.2 Multisection Quarter-Wave Transformers

Recall that a quarter-wave section of transmission line can be used as a matching network. From Eq. (1.25), the input impedance looking into a quarter-wave line ($\beta\ell = \pi/2$) is

$$Z_{in} = Z_1 \frac{Z_L + jZ_1 \tan \beta\ell}{Z_1 + jZ_L \tan \beta\ell} = \frac{Z_1^2}{Z_L} \quad (2.8)$$

where Z_1 is the characteristic impedance of the quarter-wavelength line. If we want to match a line of characteristic impedance Z_0 to the load, we must have

$$Z_0 = \frac{Z_1^2}{Z_L} \Rightarrow Z_1 = \sqrt{Z_0 Z_L} \quad (2.9)$$

What is the bandwidth of this matching network? If we change the frequency, then $\beta\ell$ is no longer equal to $\pi/2$. The reflection coefficient is

$$\Gamma = \frac{Z_{in} - Z_0}{Z_{in} + Z_0} \quad (2.10)$$

where Z_{in} is given by (1.25). Figure 2.2 shows the magnitude of the reflection coefficient as a function of frequency relative to f_0 , where f_0 is the frequency at which we have a perfect match. How can we get a broadband match?

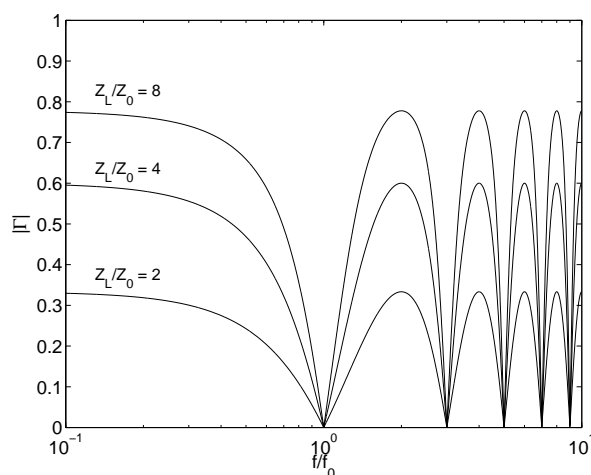


Figure 2.2: Magnitude of reflection coefficient as a function of frequency for a quarter-wavelength transformer.

2.2.1 Small Reflection Theory.

Suppose that Z_0 , Z_1 , and Z_L are close in value. If this is the case, then the reflection coefficient can be approximated by the first reflections from the two discontinuities in the transmission line:

$$\Gamma \simeq \Gamma_0 + \Gamma_1 e^{-2j\theta} \quad (2.11)$$

where

$$\begin{aligned}\Gamma_0 &= \frac{Z_1 - Z_0}{Z_1 + Z_0} \\ \Gamma_1 &= \frac{Z_L - Z_1}{Z_L + Z_1} \\ \theta &= \beta\ell\end{aligned}$$

For a multisection transformer as shown in Fig. 2.3 with sections of impedance $Z_0, Z_1, Z_2, \dots, Z_N$ followed by a load Z_L , the reflection coefficient is approximately

$$\Gamma \simeq \Gamma_0 + \Gamma_1 e^{-2j\theta} + \dots + \Gamma_N e^{-2jN\theta} \tag{2.12}$$

If we assume that the sections are symmetrical, so that $\Gamma_0 = \Gamma_N, \Gamma_1 = \Gamma_{N-1}$, and so on, then we can combine pairs of terms to get

$$\Gamma = 2e^{-jN\theta} (\Gamma_0 \cos N\theta + \Gamma_1 \cos (N-2)\theta + \dots + \Gamma_{N/2/2}) \tag{2.13}$$

for N even, with a similar expression for N odd. Notice that this is a Fourier cosine series! This means that we can get any reflection coefficient as a function of frequency we want, with enough sections and by choosing the individual reflection coefficients Γ_n properly. There are a number of different ways to accomplish this.

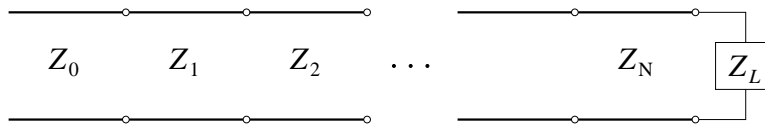


Figure 2.3: Multisection quarter wavelength transformer.

Binomial Multisection Matching. One possibility is to choose the reflection coefficient to be of the form

$$\Gamma = A(1 + e^{-2j\theta})^N \tag{2.14}$$

Since $1 + e^{-2j\theta} = 0$ for $\theta = \pi/2$ (at the center frequency), near the center frequency this quantity is small, and raising it to the N th power makes it even smaller. This means that the reflection coefficient is closer to zero over a broader band around the center frequency for large N . More rigorously, this form means that $N - 1$ derivatives of Γ at $\theta = \pi/2$ are zero.

The constant A can be found by letting the frequency go to zero in (2.14) and using the fact that at very low frequencies the sections are electrically short and have negligible effect on the reflection coefficient:

$$\Gamma(0) = 2^N A = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{2.15}$$

If we expand Γ using the binomial theorem, we get

$$\Gamma = A \binom{N}{0} + A \binom{N}{1} e^{-2j\theta} + \dots + A \binom{N}{N} e^{-2jN\theta} \tag{2.16}$$

where the binomial coefficient is

$$\binom{N}{n} = \frac{N!}{(N-n)!n!} \tag{2.17}$$

By comparing this equation to (2.12), we can read off the values of Γ_n and determine the impedances Z_1, Z_2, \dots, Z_N .

One way to get the impedances of each section from the values of Γ_n would be to use the expression for the reflection coefficient in terms of Z_n and Z_{n+1} , but if we do this for Z_2, Z_3 , and so on then Z_{N+1} will be slightly different from Z_L . If we use

$$\Gamma_n = \frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n} \simeq \frac{1}{2} \ln \frac{Z_{n+1}}{Z_n} \quad (2.18)$$

instead, we avoid this problem and get a formula for Z_n that is easy to use. From Eqs. (2.12) and (2.16), we find that

$$\begin{aligned} \ln \frac{Z_{n+1}}{Z_n} &\simeq 2A \binom{N}{n} \\ &= 2^{-N} 2 \frac{Z_L - Z_0}{Z_L + Z_0} \binom{N}{n} \\ &\simeq 2^{-N} \binom{N}{n} \ln \frac{Z_L}{Z_0} \end{aligned} \quad (2.19)$$

from which each of the impedances Z_n can be found starting with $n = 1$.