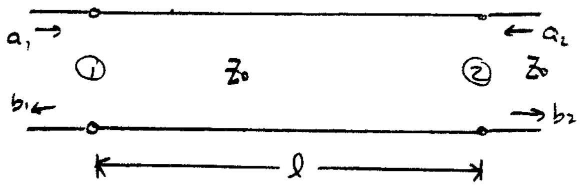
4.10 Derive the S-matrix for each of the following networks, relative to a system impedance of Z_0.



Since the line is of impedance Z_0, if we terminate port 2 in Z_0, $S_{11}=0$. Similarly, $S_{22}=0$.

On the line: $V(z) = V^+e^{-j\beta z} + V^- e^{j\beta z} = V^+e^{-j\beta z}$ since $V^-=0$ (no reflections due to match)

Since $S_{11}=0$, $b_1=0$
Also, $a_2=0$.

$$S = \boxed{\begin{bmatrix} 0 & e^{-j\beta\ell} \\ e^{-j\beta\ell} & 0 \end{bmatrix}}$$

Obviously:

$|S_{11}|^2+|S_{21}|^2 = 0+1=1$

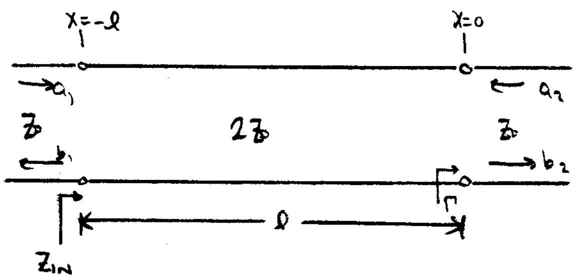
$|S_{12}|^2+|S_{22}|^2 = 1$

$S_{11}S_{12}^* + S_{21}S_{22}^* = 0$

$\sqrt{Z_0}\, a_1 = V(-\ell) = V^+e^{j\beta\ell}$

$\sqrt{Z_0}\, b_2 = V(0) = V^+$

$$\frac{b_2}{a_1} = \frac{1}{e^{j\beta\ell}} = e^{-j\beta\ell}$$



If we terminate port ② in Z_0

$$Z_{IN} = 2Z_0 \frac{Z_0 + j2Z_0\tan\beta\ell}{2Z_0 + jZ_0\tan\beta\ell} = 2Z_0\frac{1+j2\tan\beta\ell}{2+j\tan\beta\ell}$$

$$S_{11} = \frac{Z_{IN}-Z_0}{Z_{IN}+Z_0} = \frac{2Z_0(1+j2\tan\beta\ell) - Z_0(2+j\tan\beta\ell)}{2Z_0(1+j2\tan\beta\ell) + Z_0(2+j\tan\beta\ell)} = \frac{j3Z_0\tan\beta\ell}{4Z_0 + j5Z_0\tan\beta\ell} = \frac{3}{5-j4\cot\beta\ell}$$

On the line: $V(x) = V^+e^{-j\beta x} + V^-e^{j\beta x} = V^+\left[e^{-j\beta x} + \Gamma e^{j\beta x}\right]$

$$\Gamma = \frac{Z_0 - 2Z_0}{Z_0+2Z_0} = -\frac{1}{3}$$

$$V(x) = V^+\left[e^{-j\beta x} - \tfrac{1}{3}e^{j\beta x}\right]$$

Equality of voltages at $x=0$ and $-\ell$

$$V(0) = V^+\left[1-\tfrac{1}{3}\right] = \sqrt{Z_0}\,(a_2+b_2)$$

(E1) $\frac{2}{3}V^+ = \sqrt{Z_0}\, b_2$ (since $a_2=0$ when port ② is terminated in Z_0)

$$V(-\ell) = V^+\left[e^{j\beta\ell} - \tfrac{1}{3}e^{-j\beta\ell}\right] = \sqrt{Z_0}(a_1+b_1)$$

$$(E2) \qquad V^+\left[e^{j\beta\ell} - \tfrac{1}{3}e^{-j\beta\ell}\right] = \sqrt{Z_0}\,a_1(1+S_{11})$$

Taking the ratio of (E1) and (E2):

$$\frac{\sqrt{Z_0}\,b_2}{\sqrt{Z_0}\,a_1(1+S_{11})} = \frac{\tfrac{2}{3}V^+}{V^+\left[e^{j\beta\ell} - \tfrac{1}{3}e^{-j\beta\ell}\right]}$$

$$S_{21} = \frac{b_2}{a_1} = \frac{2(1+S_{11})}{3\left(e^{j\beta\ell} - \tfrac{1}{3}e^{-j\beta\ell}\right)}$$

$$\boxed{\begin{aligned} S_{11} = S_{22} &= \frac{3}{5 - j4\cot\beta\ell} \\ S_{21} = S_{12} &= \frac{2(1+S_{11})}{3\left(e^{j\beta\ell} - \tfrac{1}{3}e^{-j\beta\ell}\right)} \end{aligned}}$$

4.16

$$S = \begin{bmatrix} 0.1\angle 90^\circ & 0.8\angle -45^\circ & 0.3\angle 45^\circ & 0 \\ 0.8\angle -45^\circ & 0 & 0 & 0.4\angle 45^\circ \\ 0.3\angle -45^\circ & 0 & 0 & 0.6\angle -45^\circ \\ 0 & 0.4\angle 45^\circ & 0.6\angle -45^\circ & 0 \end{bmatrix}$$

(a) Is this network lossless?

Try: $|S_{11}|^2 + |S_{21}|^2 + |S_{31}|^2 + |S_{41}|^2 = 0.1^2 + 0.8^2 + 0.3^2 + 0 = 0.74 \neq 1$

$\boxed{\text{No.}}$

(b) Is this network reciprocal?

Since $S = S^T$ $\boxed{\text{Yes}}$

(c) What is the return loss at port 1 when all other ports are matched?

With all ports matched: $RL = -20\log|S_{11}| = \boxed{20\,dB}$

4.16 (d) What is the insertion loss and phase between ports 2 and 4, when all other ports are matched?

$$IL = -20\log|S_{42}| = -20\log 0.4 \Rightarrow \boxed{\begin{array}{l} IL = 7.96\text{ dB} \\ \text{Insertion Phase} = \angle S_{42} = 45^\circ \end{array}}$$

(e) What is the reflection coefficient at port 1 if port 3 is shorted and all other ports are matched?

$a_2 = a_4 = 0$

$a_3 = -b_3$

$$b_1 = S_{11}a_1 + S_{12}a_2 + S_{13}a_3 + S_{14}a_4 = S_{11}a_1 - S_{13}b_3 \qquad (E1)$$

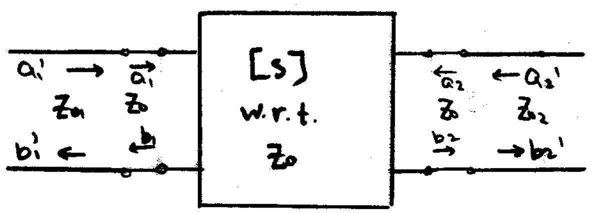
$$b_3 = S_{31}a_1 + S_{32}a_2 + S_{33}a_3 + S_{34}a_4 = S_{31}a_1 - S_{33}b_3 \qquad (E2)$$

Solving (E2): $b_3 = \dfrac{S_{31}}{1+S_{33}}\, a_1$

Placing this into (E1): $b_1 = S_{11}a_1 - \dfrac{S_{13}S_{31}}{1+S_{33}}\, a_1$

$$\boxed{\Gamma_1 = \frac{b_1}{a_1} = S_{11} - \frac{S_{13}S_{31}}{1+S_{33}} \qquad 0.1\angle 90^\circ - 0.3^2\angle -90^\circ = j(0.1+0.3^2) = 0.19\angle 90^\circ}$$

4.20 A two port has S-parameters S_{ij} for a system impedance of Z_0. Find the new S parameters for the network shown.



Assume an infinitesimal length of line (Z_0) attached to the end of the 2-port.

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

If we terminate port ② with Z_{02} ($a_2' = 0$)

$$\Gamma_2 = \frac{Z_{02} - Z_0}{Z_{02} + Z_0}, \qquad a_2 = \Gamma_2 b_2$$

$$b_2 = S_{21} a_1 + S_{22} \Gamma_2 b_2 \qquad \text{(from 2nd line of matrix)}$$

$$b_2 = \frac{S_{21}}{1 - S_{22}\Gamma_2} a_1$$

$$b_1 = S_{11} a_1 + S_{12} \Gamma_2 b_2 \qquad \text{(from 1st line of matrix)}$$

$$= S_{11} a_1 + \frac{S_{12}\Gamma_2 S_{21}}{1 - S_{22}\Gamma_2} a_1 = \left[S_{11} + \frac{S_{12} S_{21} \Gamma_2}{1 - S_{22}\Gamma_2} \right] a_1$$

So, $\dfrac{b_1}{a_1} = S_{11} + \dfrac{S_{12} S_{21} \Gamma_2}{1 - S_{22}\Gamma_2} = \Gamma_{11}$ is the reflection coefficient with respect to Z_0.

We need to compute the reflection coefficient with respect to Z_{01}.

$$Z_{11} = Z_0 \frac{(1 + \Gamma_{11})}{(1 - \Gamma_{11})}$$

$$S_{11}' = \frac{Z_{11} - Z_{01}}{Z_{11} + Z_{01}} = \frac{Z_0(1 + \Gamma_{11}) - Z_{01}(1 - \Gamma_{11})}{Z_0(1 + \Gamma_{11}) + Z_{01}(1 - \Gamma_{11})}$$

$$\boxed{S_{11}' = \frac{Z_0(1 + \Gamma_{11}) - Z_{01}(1 - \Gamma_{11})}{Z_0(1 + \Gamma_{11}) + Z_{01}(1 - \Gamma_{11})} \quad \text{where} \quad \Gamma_{11} = S_{11} + \frac{S_{12} S_{21} \Gamma_2}{1 - S_{22}\Gamma_2}, \quad \Gamma_2 = \frac{Z_{02} - Z_0}{Z_{02} + Z_0}}$$

4.20 (cont)

Now: $\sqrt{Z_0}(a_1+b_1) = \sqrt{Z_{01}}(a_1'+b_1')$

$$\sqrt{Z_0}a_1(1+\Gamma_{11}) = \sqrt{Z_{01}}a_1'(1+S_{11}') \quad (1)$$

Also: $\sqrt{Z_0}(a_2+b_2) = \sqrt{Z_{02}}b_2'$

$$\sqrt{Z_0}b_2(\Gamma_2+1) = \sqrt{Z_{02}}b_2' \quad (2)$$

Taking the ratio of (2) to (1) leads to:

$$\frac{\sqrt{Z_{02}}\,b_2'}{\sqrt{Z_{01}}\,a_1'(1+S_{11}')} = \frac{\sqrt{Z_0}\,b_2(1+\Gamma_2)}{\sqrt{Z_0}\,a_1\,(1+\Gamma_{11})}$$

$$S_{21}' = \frac{b_2'}{a_1'} = \frac{\sqrt{Z_{01}}}{\sqrt{Z_{02}}}\,\frac{(1+S_{11}')(1+\Gamma_2)}{(1+\Gamma_{11})}\,\frac{b_2}{a_1}$$

$$\boxed{S_{21}' = \frac{\sqrt{Z_{01}}}{\sqrt{Z_{02}}}\,\frac{(1+S_{11}')(1+\Gamma_2)}{(1+\Gamma_{11})}\,\frac{S_{21}}{1-S_{22}\Gamma_2}}$$

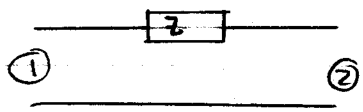
We can perform the same steps for terminating port (1) and then injecting the signal into port (2). In the mathematics, however, we just interchange the subscripts 1 and 2

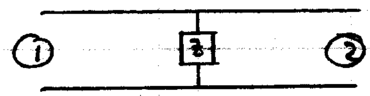
$$S_{22}' = \frac{Z_0(1+\Gamma_{22}) - Z_{02}(1-\Gamma_{22})}{Z_0(1+\Gamma_{22}) + Z_{02}(1-\Gamma_{22})} \quad \text{where} \quad \Gamma_{22} = S_{22} + \frac{S_{12}S_{21}\Gamma_1}{1-S_{11}\Gamma_1}$$

$$\Gamma_1 = \frac{Z_{01}-Z_0}{Z_{01}+Z_0}$$

$$S_{12}' = \frac{\sqrt{Z_{02}}}{\sqrt{Z_{01}}}\,\frac{(1+S_{22}')(1+\Gamma_1)}{(1+\Gamma_{22})}\,\frac{S_{12}}{1-S_{11}\Gamma_1}$$

4.28 Find the S-parameters. Show that $S_{12} = 1 - S_{11}$ for the series case, and $S_{12} = 1 + S_{11}$ for the shunt case. Assume a characteristic impedance of Z_0.





Series: For S_{11}, terminate port 2 in Z_0.

$$Z_{IN} = Z + Z_0$$

$$S_{11} = \frac{Z + Z_0 - Z_0}{Z + Z_0 + Z_0} = \frac{Z}{Z + 2Z_0}$$

$$V_1 = \sqrt{Z_0}(a_1 + b_1) = \sqrt{Z_0}\, a_1 (1 + S_{11})$$
$$V_2 = \sqrt{Z_0}\, b_2$$

But $\dfrac{V_2}{V_1} = \dfrac{Z_0}{Z + Z_0} = \dfrac{\sqrt{Z_0}\, b_2}{\sqrt{Z_0}\, a_1 (1 + S_{11})}$

$$S_{21} = \frac{b_2}{a_1} = (1 + S_{11})\frac{Z_0}{Z + Z_0} = \left(1 + \frac{Z}{Z + 2Z_0}\right)\left(\frac{Z_0}{Z + Z_0}\right) = \frac{Z + 2Z_0 + Z}{Z + 2Z_0}\,\frac{Z_0}{Z + Z_0}$$

$$= \frac{2(Z + Z_0)}{Z + 2Z_0}\,\frac{Z_0}{Z + Z_0} = \frac{2Z_0}{Z + 2Z_0}$$

So: $S_{11} = S_{22} = \dfrac{Z}{Z + 2Z_0}$

$$S_{21} = S_{12} = \frac{2Z_0}{Z + 2Z_0}$$

Note that $1 - S_{11} = \dfrac{Z + 2Z_0 - Z}{Z + 2Z_0} = \dfrac{2Z_0}{Z + 2Z_0} = S_{12}$

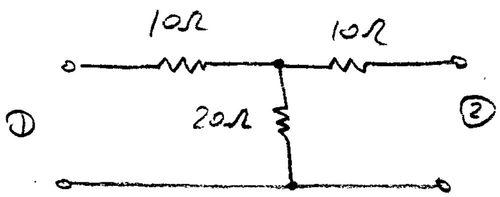
Shunt: $Y_{IN} = \dfrac{1}{Z} + \dfrac{1}{Z_0} = Y + Y_0$

$$S_{11} = \frac{Y_0 - Y_{IN}}{Y_0 + Y_{IN}} = \frac{Y_0 - Y - Y_0}{Y_0 + Y + Y_0} = \frac{-Y}{Y + 2Y_0} = S_{22}$$

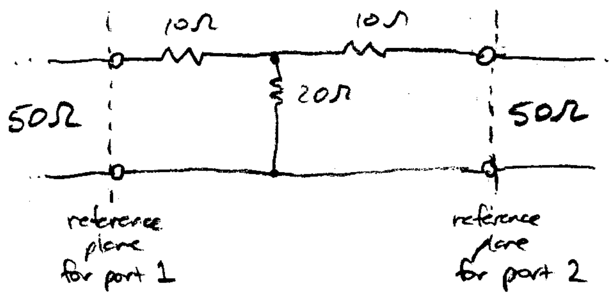
$$\left.\begin{aligned} V_1 &= \sqrt{Z_0}\, a_1 (1 + S_{11}) \\ V_2 &= \sqrt{Z_0}\, b_2 = V_1 \end{aligned}\right\} \Rightarrow \frac{b_2}{a_1(1 + S_{11})} = 1$$

$S_{21} = 1 + S_{11} = \dfrac{2Y_0}{Y + 2Y_0} = S_{12}$ which also proves the requested relation.

Find the S parameters of this T network:



Since the system impedance is not specified, we use 50Ω:

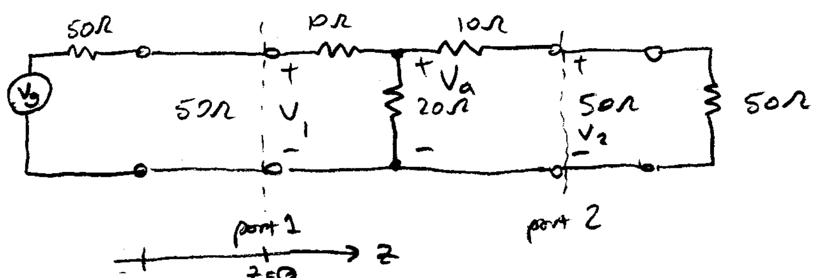


By symmetry, $S_{21} = S_{12}$ and $S_{11} = S_{22}$.

$$Z_{IN\,1} = 10\Omega + 20\Omega \,||\, (10\Omega + 50\Omega) = 25$$

$$S_{11} = \Gamma_{in_1} = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{25-50}{25+50} = -\frac{1}{3}$$

To find S_{21}, apply a source to port 1 and match port 2:



On the "terminal" line connected to port 1,

$$V(z) = V_1^+ e^{-j\beta z} + V_1^- e^{j\beta z}$$

$$= V_1^+ e^{-j\beta z} + \Gamma_{in_1} V_1^+ e^{j\beta z}$$

$$V(0) = V_1^+ (1 + \Gamma_{in\,1}) = V_1^+ (2/3) = V_1 \Rightarrow V_1^+ = \tfrac{3}{2} V_1$$

At port 2,

$V_2 = V_2^-$ (since there is no reflected wave on the terminal line connected to port 2)

460

$$V_a = V_1 \cdot \frac{20 \parallel (10+50)}{10 + 20 \parallel (10+50)} = \frac{3}{5} V_1 \qquad \text{(voltage divider)}$$

$$V_2 = V_a \cdot \frac{50}{10+50} = \frac{5}{6} V_a \qquad \text{(voltage divider)}$$

so, $$V_2 = \frac{5}{6} \cdot \frac{3}{5} V_1 = \frac{1}{2} V_1$$

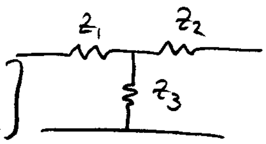
and $$S_{21} = \frac{V_2^-}{V_1^+} = \frac{V_2}{V_1^+} = \frac{1/2\, V_1}{3/2\, V_1} = \underline{\underline{\frac{1}{3}}}.$$

Thus, $$\bar{\bar{S}} = \begin{bmatrix} -1/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix}.$$

<u>Another method</u>

From p. 208, Pozar, the transmission matrix is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 + Z_1/Z_3 & Z_1 + Z_2 + Z_1 Z_2 / Z_3 \\ 1/Z_3 & 1 + Z_2/Z_3 \end{bmatrix} = \begin{bmatrix} 3/2 & 25 \\ 1/20 & 3/2 \end{bmatrix}$$



From p. 211, the S parameters are

$$S_{11} = \frac{A + B/Z_0 - CZ_0 - D}{A + B/Z_0 + CZ_0 + D} = \frac{-2}{6} = \underline{\underline{-\frac{1}{3}}}$$

$$S_{21} = \frac{2}{A + B/Z_0 + CZ_0 + D} = \frac{2}{6} = \underline{\underline{\frac{1}{3}}}$$